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# Universal fluctuations of zeros of chaotic wavefunctions

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**Abstract.** Wavefunctions of one and two-dimensional quantum systems can be parametrized by a finite number of zeros lying in phase space. We study correlations of these zeros for fully chaotic systems in terms of a statistical model based on random polynomials. Excellent agreement is found for the two-point correlation function and nearest-neighbour spacing distribution of this model and the results obtained for wavefunctions of dynamical systems. We conjecture that these correlation functions are valid for any chaotic system after rescaling the phase-space distances (unfolding). Some consequences for the distribution of zeros due to time-reversal symmetry are also discussed.

## 1. Introduction

The statistical properties of the quantum eigenvalues of classically chaotic systems are known to be, in the universal regime, in agreement with the results of the appropriate symmetry class of random matrix ensembles (see e.g. [1, 2]). One basic feature emerging from these ensembles is the repulsion between energy levels. This repulsion can be interpreted as an electrostatic repulsion, since the joint probability density of the eigenvalues of Gaussian ensembles can be interpreted as the probability density for the positions of a one-dimensional gas of charged particles interacting via a two-dimensional Coulomb force [3]. The Coulomb gas model was moreover extended to non-equilibrium ensembles, the so-called Dyson's Brownian motion model (see ch 8 in [1]), which has been useful in the study of the transition between different symmetry classes.

In contrast, the wavefunctions of quantum chaotic systems have so far remained a relatively less explored (as compared to spectra) area, notwithstanding their importance in quantum mechanical as well as semiclassical analysis. Studies of statistical properties of wavefunctions of chaotic systems have been done in the past for two-dimensional chaotic billiards [4] or mappings [5], as well as experimental tests [6]. These are mainly concentrated on the validity of the Porter–Thomas distribution and the spatial autocorrelation function. The former is a limited test of the full distribution of the wavefunction (see equation (4) below), since it is equivalent to the fact that the distribution of one amplitude $a_k$ is Gaussian. In this paper our aim is to study the statistical distribution of zeros of 'quantum chaotic' wavefunctions expressed in phase-space representation. The interest in the phase space study of nodal patterns arises following a study by Lebœuf and Voros [7] where they show that the coherent state representation of wavefunctions of one-dimensional systems (i.e. Bargmann

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or Husimi functions) have a finite number $N$ of zeros ($N$ being proportional to the dimension of Hilbert space) and the quantum state is uniquely defined, up to a normalization constant, by their positions. This scheme can be easily extended to two dimensions [8]. Furthermore, their numerical studies on quantum maps (e.g. Baker and kicked rotor) have also revealed that, in the semiclassical regime $\overline{h} \to 0$, the zeros condense on lines for classically integrable systems whereas for strongly chaotic quantum maps they appear to diffuse fairly uniformly over the phase space. Thus, the direct manifestation of classical motion in the distribution of zeros makes it relevant to seek further information about their behaviour, in particular about their correlations.

As we will see in the next section, assuming that the amplitudes of a quantum eigenstate have a Gaussian independent distribution (cf equation (4) below) and, moreover, considering a coherent state representation of quantum mechanics, we end up with a statistical model for the study of wavefunctions of chaotic systems in terms of random polynomials. From this model, which was introduced in [9], we compute the joint probability density for the zeros of wavefunctions. This probability density can be interpreted as the probability density of a two-dimensional gas of $N$ interacting particles lying in phase space. Aside from a two-dimensional Coulomb force, there are also $n$-body interactions, with $3 \leqslant n \leqslant N$. In section 3 the nearest-neighbour spacing distribution and the two-point correlation function are studied and compared to results obtained from different physical and/or mathematical models. Our main conclusions are that there exists a strong repulsion between zeros which makes the gas of zeros quite rigid (quadratic repulsion at short distances) and that the results obtained from the model of random polynomials are in very good agreement with those originated in quantum chaotic systems. Finally, section 4 deals with some consequences for the distribution of zeros due to a time reversal symmetry. While this work was in progress, we learned of an analytical computation of correlations of zeros by Hannay [10]. We have included his result concerning the two-point function in section 3.2.

## 2. The joint probability density

For definiteness, let us for the moment consider the case of a spin $\boldsymbol{J}$ system, whose modulus $J$ is preserved by the dynamics (different geometries will be considered later). Classically the motion of the arrow $\boldsymbol{J}$ in the three-dimensional space can be represented by a point moving on the surface of a two-dimensional (Riemann) sphere, denoted $\mathcal{S}$, which is the phase space. Quantum mechanically $J$ can only take the values $\frac{1}{2}, 1, \ldots$ , while the Hilbert space is finite and $(2J+1)$-dimensional. An arbitrary quantum state of the spin projected onto spin (or $SU(2)$) coherent-states $|z\rangle$ can be written [11, 12]

$$\psi(z) = \sum_{k=0}^{N} \sqrt{C_N^k}\, a_k z^k \tag{1}$$

where $N = 2J$ and $C_N^k$ are the binomial coefficients. We assume for the moment that the system has no time-reversal symmetry (see however section 4), and hence the amplitudes $\{a_k\}$ are complex. For convenience the radius of the sphere, given by $\overline{h}\sqrt{J(J+1)}$, is normalized to one. Then the classical limit of such models corresponds to $N = 2J \to \infty$. Moreover the complex variable $z$ labelling the coherent states and appearing in equation (1) is connected to the variables $(\theta, \varphi)$ spanning the Riemann sphere by a stereographic projection of the plane onto the sphere given by $z = \cot(\theta/2)\mathrm{e}^{\mathrm{i}\varphi}$. The function $\psi(z)$ is therefore an analytic function defined on the two-dimensional sphere. The normalization

in the space of functions is given by

$$\sum_{k=0}^{N} |a_k|^2 = \int \mathrm{d}^2 z\, \mu(z,\bar{z}) |\psi(z)|^2$$

where the measure is $\mu(z,\bar{z}) = (N+1)/[\pi(1+|z|^2)^{(N+2)}]$. Because $\psi(z)$ is a polynomial of degree $N$, it has $N$ zeros $\boldsymbol{z} = (z_1, z_2, \ldots, z_N)$ in that space which completely determine, up to a global normalization factor, the quantum state

$$\psi(z) = a_N \prod_{k=1}^{N} (z - z_k). \tag{2}$$

Now that we have introduced the framework let us concentrate on statistical properties. Assume that $\psi(z)$ given by (1) is an eigenstate of a chaotic system. There is a prediction of the random matrix theory (RMT) for the amplitudes $\{a_k\}$ which follows directly from the symmetries of the ensemble. Assuming there is no time-reversal symmetry, the invariance under unitary transformations of the Gaussian unitary ensemble implies that the joint distribution function of the complex amplitudes $\boldsymbol{a} = (a_0, a_1, \ldots, a_N)$ is given by [6]

$$\mathcal{D}_{RMT}(\boldsymbol{a}) = \frac{1}{|S_{2(N+1)}|} \delta\left[1 - \sum_{k=0}^{N} |a_k|^2\right] \tag{3}$$

where $|S_n| = 2\pi^{n/2}/\Gamma(n/2)$ is the surface of an $(n-1)$-dimensional sphere of unit radius. If we are concerned by average properties of functions that depend only on the zeros of $\psi(z)$, the distribution (3) for the coefficients is strictly equivalent to

$$\mathcal{D}(\boldsymbol{a}) = \frac{1}{(2\pi)^{(N+1)}} \exp\left\{ -\frac{1}{2} \sum_{k=0}^{N} |a_k|^2 \right\}. \tag{4}$$

This is because, as shown on p 6 of [13], both averages are equal when computed over a function which does not depend on the modulus of $\boldsymbol{a}$. Thus, for our purposes, the real and imaginary part of the amplitudes $a_k$ can be considered as Gaussian uncorrelated random variables.

From (4) it is easy to compute the joint probability distribution for the zeros [9, 14, 10]. The change of variables $\mathcal{D}(\boldsymbol{a})\,\mathrm{d}^2 a_0 \ldots \mathrm{d}^2 a_N = \mathcal{D}(\boldsymbol{z})\,\mathrm{d}^2 z_1 \ldots \mathrm{d}^2 z_N$ (where $\mathrm{d}^2 x \equiv \mathrm{d}(\mathrm{Re}\, x)\,\mathrm{d}(\mathrm{Im}\, x)$) leads to

$$\mathcal{D}(\boldsymbol{z}) = \frac{N!}{\pi^N \prod_{k=0}^{N} C_N^k} \left\{ \prod_{i<j} |z_i - z_j|^2 \left[ \left( 1 + \frac{1}{C_N^{N-1}} |z_1 + z_2 + \cdots|^2 + \frac{1}{C_N^{N-2}} |z_1 z_2 + z_1 z_3 + \cdots|^2 + \cdots + \frac{1}{C_N^0} |z_1 z_2 \cdots z_N|^2 \right)^{N+1} \right]^{-1} \right\}. \tag{5}$$

If this probability density is interpreted as the Boltzmann factor of a partition function then the set of zeros can be thought of as a two-dimensional gas of interacting particles. The numerator of (5), coming from the Jacobian of the transformation, contributes with a factor proportional to a two-dimensional Coulomb potential $\ln|z_i - z_j|$. The denominator, however, contributes with terms that can be interpreted as $n$-body interactions, with $3 \leqslant n \leqslant N$ [15]. The function $\mathcal{D}(\boldsymbol{z})$ has also been re-expressed and interpreted geometrically in terms of the relative coordinates of the zeros on the sphere [10]. Moreover, for this ensemble it was shown [9, 14, 10] that for arbitrary $N$ the density of zeros is uniform on the Riemann sphere (a sort of ergodicity of wavefunctions).

It is interesting to compare (5) with a different and well known two-dimensional distribution of points on the plane, defined by the joint probability density for the eigenvalues

$z_i$ of a Gaussian ensemble of $N \times N$ complex matrices obtained by Ginibre [16]

$$\mathcal{D}_{Gin}(\boldsymbol{z}) = \mathcal{N} \exp\left( -\sum_{k=1}^{N} |z_k|^2 \right) \prod_{i<j} |z_i - z_j|^2 \tag{6}$$

which can be physically interpreted as a two-dimensional one-component plasma [17]. In this case the density is uniform in the $z$-plane $z = x + \mathrm{i}y$ inside a disk of radius $r \sim \sqrt{N}$ [1], and zero outside.

## 3. Correlations

Our purpose in this section is, on the one hand, to understand the nature of the correlations between zeros and on the other to explore to what extent the predictions of the model (5) agree with the results obtained for wavefunctions of classically chaotic systems. As a model of a chaotic system, we have chosen a kicked top map [5], defined by the unitary operator

$$U = \mathrm{e}^{-\frac{\mathrm{i}\nu}{2\hbar} J_z^3} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\mu J_x} \mathrm{e}^{-\frac{\mathrm{i}}{2\hbar} p J_z^2} \tag{7}$$

acting on the $(2J+1)$-dimensional Hilbert space. The eigenfunctions are determined by the stationary equation

$$U|\psi_\alpha >= \mathrm{e}^{\mathrm{i}\omega_\alpha}|\psi_\alpha\rangle \qquad \alpha = 1, \ldots, 2J+1$$

which was solved, for all the numerical computations of correlations quoted below, for $J = 49$. The matrix $U$ was diagonalized for 294 different sets of parameters centred around $p = 17$, $\mu = 2$, $\nu = 6$ for which the classical dynamics of the map is dominated by chaotic trajectories. Correlations of zeros were computed for each eigenfunction and all the results of the $99 \times 294$ eigenfunctions superimposed in order to have better statistics.

### *3.1. Nearest-neighbour spacing distribution*

We have computed numerically the nearest-neighbour spacing distribution $P(r)$ between zeros, shown in figure 1: this quantity is defined as the probability distribution for the distance between a zero and its nearest neighbour. The distribution was normalized with the average number of points lying at a distance $r$ from a reference point, $\sin(\Delta r)$, with $r$ being the arc length (or relative angle) on the sphere measured in units of the mean spacing $\Delta = \sqrt{4\pi/N}$ between zeros. The broken curve is the result obtained from 29 106 different

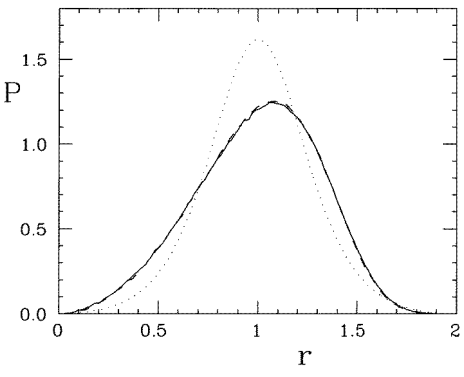


**Figure 1.** Nearest-neighbour spacing distribution. Full curve: zeros of eigenstates of the kicked top map (7). Broken curve: zeros of random polynomials. Dotted line: formula (8) ($N = 2$).

trials of the random polynomial (1) for $N = 98$ with the distribution (4) for the coefficients. This gives the same total number of zeros as for the kicked top model (7), represented in the figure by a full curve. The agreement between the correlations for the dynamical system and those of random polynomials is excellent, and the two curves are almost indistinguishable.

In the absence of an analytical solution for the nearest-neighbour spacing distribution at arbitrary (or asymptotic) $N$, we may compute the spacing distribution for $N = 2$. This would be the analogue of the Wigner surmise for the nearest-neighbour spacing distribution of eigenvalues of random matrices, computed for $2 \times 2$ matrices. From (5) for $N = 2$ and changing variables we find

$$\mathcal{D}_2(\theta_1, \varphi_1, \theta_2, \varphi_2) = \frac{2}{\pi^2} \frac{(1 - \cos\xi)}{(3 + \cos\xi)^3}.$$

where $\xi$ is the relative angle between zeros, $\cos\xi = \sin\theta_1 \sin\theta_2 \cos(\varphi_1 - \varphi_2) + \cos\theta_1 \cos\theta_2$. The interpretation of this probability density as the Boltzmann factor of a partition function $\exp(-\beta V)$ gives two terms for the potential. As already mentioned, the first one (coming from the numerator) can be interpreted as a two-dimensional Coulomb repulsion with a distance proportional to the chord length on the sphere, while the second one is also repulsive but with a force vanishing at zero distance.

The probability density of the spacing $r$ between the two zeros may be obtained by integration. Normalizing by the Jacobian, we finally find

$$P_2(r) = \frac{32\sqrt{2}}{7} \frac{[1 - \cos(\pi r)]}{[3 + \cos(\pi r)]^3} \tag{8}$$

which satisfies the normalization conditions $\int_0^2 P_2(r)\,\mathrm{d}r = \int_0^2 r\,P_2(r)\,\mathrm{d}r = 1$. For short distances $P_2(r) \simeq \pi^2 r^2/(14\sqrt{2}) + \mathcal{O}(r^4)$, which reveals a strong repulsion (see below). The coefficient of the quadratic term is, however, too small since the exact one, computed in the next subsection, is $\pi/2$ (cf equation (10)). The disagreement between $P_2(r)$ and $P(r)$ computed for large values of $N$ is not limited to short distances. The curve (8), represented in figure 1 by a dotted line, is not a good approximation to $P(r)$ for any value of $r$.

For comparison with the previous results, we quote here the nearest-neighbour spacing distribution for uncorrelated points thrown at random on a two-dimensional plane

$$\tilde{P}_u(r) = \frac{\pi}{2} r \exp\left(-\frac{\pi}{4} r^2\right)$$

which becomes, dividing out the geometrical factor and normalizing

$$P_u(r) = \mathcal{N} \frac{\tilde{P}_u(r)}{r} = \frac{2}{\pi} \exp\left(-\frac{1}{\pi} r^2\right)$$

which has no repulsion at short distances.

### *3.2. Two-point correlation function*

The two-point correlation function $R_2(\boldsymbol{r}_1, \boldsymbol{r}_2) = \langle \rho(\boldsymbol{r}_1)\rho(\boldsymbol{r}_2) \rangle$ (and more generally the $k$-point correlation function) for the joint probability density (5) was recently computed analytically by Hannay [10]. Here $\rho$ is the density of zeros, $(\boldsymbol{r}_1, \boldsymbol{r}_2)$ are two points on the sphere and the brackets denote ensemble average. Because the density is constant, $R_2$ depends only on a relative coordinate. In the asymptotic large-$N$ limit it is given by

$$R_2(r) \simeq \frac{[(\sinh^2 v + v^2)\cosh v - 2v \sinh v]}{\sinh^3 v} \tag{9}$$

where $v = \pi r^2/2$ and $r$ is, as in the previous subsection, the arc length separating two zeros on the sphere normalized to the mean spacing $\Delta = \sqrt{4\pi/N}$. Again, as for the spacing

distribution, geometrical factors have been divided out. This function has the limiting behaviour

$$\begin{aligned} R_2(r) &\simeq \frac{\pi}{2} r^2 + \mathcal{O}(r^6) \qquad r \to 0 \\ R_2(r) &\simeq 1 + \pi^2 r^4 \mathrm{e}^{-\pi r^2} + \mathcal{O}(r^2 \mathrm{e}^{-\pi r^2}) \qquad r \to \infty. \end{aligned} \tag{10}$$

We have numerically computed this quantity for the kicked top map, shown in figure 2 by a full curve. In the same figure, equation (9) is represented by a broken curve. Again, the agreement is excellent for all $r$, and the two curves are so close to each other that they are indistinguishable in the figure. Notice the bump around $r = 1$ that characterizes the curve. The inset shows a close view of the behaviour at the origin.

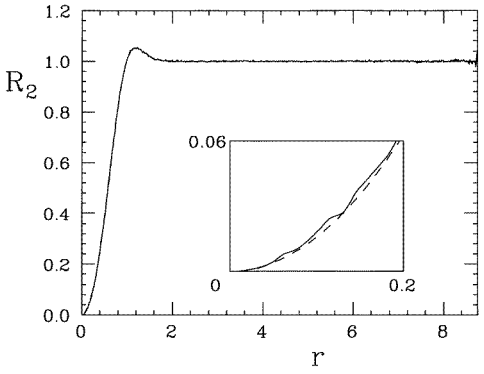


**Figure 2.** Two-point correlation function. Full curve: zeros of eigenstates of the kicked top map. Broken curve: the curve (9) computed for zeros of random polynomials. (The two curves are almost indistinguishable). The inset shows a close view of the behaviour at the origin.

These results can be compared to other statistical models. Using the same normalization, for the set of uncorrelated random points quoted above we have $R_2^u(r) = 1$, $\forall r$ . It is also interesting to compare with the two-point correlation function for Ginibre's ensemble (6),

$$R_2^{Gin}(r) = 1 - \mathrm{e}^{-\pi r^2}$$

which has no bump and behaves at the origin as $\pi r^2$.

### *3.3. Universality*

The maximum value of $r$ plotted in figure 2 is 8.75, which corresponds to $\pi$ in non-normalized units: $\pi\sqrt{N/4\pi} \simeq 8.75$ for $N = 98$. Figure 2 then covers the whole range of possible distances that can occur between two zeros on the sphere, and in all that range the agreement with the model of random polynomials is excellent. This differs from what happens for spectral statistics [18], where very long range correlations are non-universal, i.e. system-dependent. Whether this full agreement with the model of random polynomials is a special feature of the mapping considered, or if non-universalities appear in higher correlation functions, are questions that deserve further investigation.

It could be argued that the correlations obtained in the present paper are only applicable to spin systems, since they were obtained for a special class of random polynomials. However, as $N \to \infty$ the mean spacing between zeros tends to zero, and for sufficiently small distances the Riemann sphere can be considered, *locally*, as a flat space. And indeed in equation (9) the distance $r$ can equally be considered as the arc length or the chord

length separating two zeros (the expression (9) converges so fast to 1 as $r$ increases that even for large distances compared to the mean spacing the error made by using the chord length instead of the arc length is negligible). Because of these arguments, we expect the correlation functions computed from the model (5) to be *universal*, i.e. *applicable to any chaotic system, independently of the geometry of phase space*.

To check this conjecture let us consider for example a chaotic map on the two-dimensional torus, the well known kicked rotor [19]. It is defined by the unitary operator

$$U = \mathrm{e}^{-\frac{\mathrm{i}}{4\hbar}(p+\gamma_1)^2}\mathrm{e}^{-\frac{\mathrm{i}K}{4\pi^2\hbar}\cos(2\pi q+\gamma_2)}\mathrm{e}^{-\frac{\mathrm{i}}{4\hbar}(p+\gamma_1)^2} \tag{11}$$

acting on an $N$-dimensional Hilbert space of periodic functions (up to a phase). Here $q$ and $p$ are the usual position and momentum operators, and $\hbar = 2\pi/N$. The parametrization of wavefunctions in terms of zeros for systems defined on the torus was considered in [7], and the reader is referred to that reference for details. Figures 3($a$) and ($b$) show, respectively, the two-point correlation function and the nearest-neighbour spacing distribution obtained for $N = 99$, $K = 20\,000$ (fully chaotic regime) and $\gamma_1 = 0.7071$, $\gamma_2 = \pi/2N$. No average over the parameters was done in this case, and accordingly the statistical errors are bigger than for the kicked top. In figure 3($a$) we have also plotted the curve (9) and in figure 3($b$) the curve obtained numerically from the polynomial (1) with random coefficients (the same as in figure 1). The agreement is very good, and supports our conjecture. Though this test was done for a mapping, there is no reason to limit the conjecture to one-dimensional (time-dependent) systems. And indeed, some recent computations confirm the universality of correlations of zeros of wavefunctions in two-dimensional chaotic systems [20].

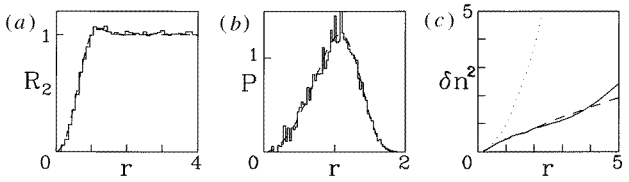


**Figure 3.** Statistical properties of the zeros of eigenstates of the kicked rotor (11) (histogram). The broken curves are the results for random polynomials (same as in figures 1 and 2). ($a$) Two-point correlation function. ($b$) Nearest-neighbour spacing distribution. ($c$) Number variance.

We have also computed for the kicked rotor the number variance, defined as the fluctuation in the number of zeros $n(r)$ inside a disk of radius $r$, $\delta n^2 = \bar{n^2} - \bar{n}^2$. This quantity is the two-dimensional analogue of the number variance quite often used in spectral statistics. For uncorrelated random points the variance increases as $r^2$. Figure 3($c$) shows that for large $r$ the growth of the number variance, for the model of random polynomials as well as for chaotic dynamical systems, is much slower than that. This long-range order reflects the strong rigidity of the gas (5) due to the repulsion between zeros.

## 4. Time-reversal symmetry

When a chaotic system has a time-reversal symmetry, instead of GUE the appropriate ensemble of random matrices is the Gaussian orthogonal. This is so because the wavefunctions can now be chosen real, and orthogonal transformations in Hilbert space replace unitary transformations. As a consequence the repulsion between energy levels is linear instead of quadratic [1].

Concerning wavefunctions, it is natural to ask what are the changes in the distribution of zeros induced by a time-reversal symmetry, and in particular how the correlations are modified.

The appropriate model to consider these questions is a set of random polynomials with real coefficients. We hence consider polynomials of the form (1) with a distribution (4) for the coefficients but with a real $\boldsymbol{a}$ vector. There is an immediate consequence of the latter condition: if $z_k$ is a zero of $\psi(z)$, then $\bar{z}_k$ is also. This means that the zeros either come by pairs symmetric with respect to the real axis or they are single and real. This leads to the question of how many real zeros this set of random polynomials has, on average. The answer, found recently by Edelman and Kostlan [21], is $\sqrt{N}$. Although the fraction of real roots vanishes in the semiclassical regime, the appearance of a privileged axis in phase space implies that the uniformity of the density of zeros valid for the model (5) is lost.

Aside from the density, the correlations are also modified. Although we have not computed them analytically for arbitrary $N$, let us illustrate this point with a simple example. We again consider the case $N = 2$ and compute the Jacobian of the transformation when the coefficients are real. The zeros are related to the coefficients of the polynomial via

$$\begin{aligned} a_1/a_2 &= -(z_1 + z_2)/\sqrt{2} \\ a_0/a_2 &= z_1 z_2. \end{aligned} \tag{12}$$

Because the left-hand side of these equations is real, there are two different solutions corresponding to the two possibilities mentioned above:

(i)

$$\begin{aligned} y_1 &= y_2 = 0 \\ a_1/a_2 &= -(x_1 + x_2)/\sqrt{2} \\ a_0/a_2 &= x_1 x_2 \end{aligned} \tag{13}$$

and (ii)

$$\begin{aligned} x_1 &= x_2 \qquad y_1 = -y_2 \\ a_1/a_2 &= -\sqrt{2}x_1 \\ a_0/a_2 &= x_1^2 + y_1^2 \end{aligned} \tag{14}$$

where we have introduced the notation $z = x + \mathrm{i}y$. The Jacobian for case (i) is $\mathcal{J} \propto |x_2 - x_1|$, i.e. real roots repel linearly. For case (ii), the Jacobian is $\mathcal{J} \propto |y_1|$, meaning that even when they are complex, but lying on different sides of the symmetry line, the zeros repel linearly. This is different from the quadratic repulsion found in previous sections in the case of complex coefficients. It also means that the density of complex zeros decreases linearly when the symmetry line is approached (aside from the $\sqrt{N}$-concentration of zeros at $y = 0$), thus creating a 'hole' in the density around that line. This effect was observed numerically in previous papers [9, 14]. Moreover, we can also test the repulsion between complex roots lying on the same side of the symmetry line. For that purpose, we have computed the Jacobian for $N = 4$, written in terms of the coordinates of the zeros $z_1$ and $z_2$ lying on one side, the other two being symmetric. We found $\mathcal{J} \propto |z_1 - z_2|^2$ (with a complicated prefactor depending on the position), i.e. like in the case of complex coefficients. Although we have not checked it numerically, we suspect that the correlations of zeros lying on the same side and sufficiently far from the symmetry line coincide in general with those obtained for complex coefficients.

## 5. Conclusion

Because the results concerning the density of zeros and their correlations depend on the Gaussian nature of the distribution as well as on their independence, the results presented here go beyond a simple verification of the Porter–Thomas distribution (or its complex analogue).

The correlations obtained from the spin model (equation (5)) are expected to hold for the quantization of any chaotic mapping (as verified here for the kicked rotor (figure 3)) but also in higher dimensional dynamical systems. The parametrization of wavefunctions by their zeros can be adapted to two-dimensional systems via Poincaré surface-of-section methods [8], and some recent computations of correlations support their universality [20]. However, for more than two dimensions the parametrization ceases to be valid.

One may expect, as for the spectral statistics, that corrections coming from short periodic orbits manifest themselves in the statistical properties of wavefunctions. For the nearest-neighbour spacing distribution and the two-point correlation function studied in this paper no traces of non-universality were found. More detailed work as well as a study of higher correlation functions can help in elucidating this question.

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